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MODELING, IDENTIFICATION AND PREDICTION OF A CLASS OF NONLINEAR VISCOELASTIC MATERIALS (I)

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Abstract—In this paper a nonlinear Volterra integral equation is used to characterize a class of nonlinear viscoelastic materials subject to uniaxial effects. A number of unknown parameters appearing in the mathematical structure of the model is optimally determined such as to minimize a least squares functional of all the experimental data, assumed to be given in terms of relaxation tests performed at various strain levels. The predictive ability of the model is tested using the experimental data furnished in [10]. The results are finally compared with those reported by Findley and Lai using a multiple integral model [9], [10] and with the experimental ones.

1. INTRODUCTION

IN A number of recent publications [1] to [6], we have systematically dealt with a number of modeling and identification problems in the realm of the theory of viscoelastic materials. In particular, in [5] we have developed a numerical procedure for the identification of nonlinear materials governed by Volterra integral equations of the form

$$\sigma = g(\varepsilon) + \int_0^t h(\varepsilon(\tau)) f(t-\tau)\,\mathrm{d}\tau,$$

from general input–output measurements. In this paper we specialize the foregoing results to study the prediction ability of such a model in the case where the material constants are to be determined exclusively from experimental data obtained from relaxation tests at various constant strain levels. In addition to the identification aspects, the numerical procedures involved in the prediction of the stresses (strains) of the specimen subject to variable strain (stress) histories are presented and thoroughly discussed. The predictive ability of the model is tested using the experimental data furnished in [10]. The results are finally compared with those reported by Findley and Lai [10], using a multiple integral model and with the experimental ones.

2. SYSTEM IDENTIFICATION

Consider a nonlinear viscoelastic material whose stress–strain relationship is given by the nonlinear integral Volterra equation

$$\sigma = g(\varepsilon(t), k_1, k_2, \ldots) + \int_0^t h(\varepsilon(\tau), d_1, d_2, \ldots) f(t-\tau)\,\mathrm{d}\tau, \tag{1}$$

where σ and ε are scalar components of the stress and strain respectively, measured on a test specimen subject to uniaxial tension or compression. The function $g(\varepsilon, k_1, k_2, \ldots)$

denotes the instantaneous nonlinear elastic response, given in parametric form in terms of unknown constants $k_1, k_2, \ldots$, whereas $h(\varepsilon, d_1, d_2, \ldots)$ will be in general a nonlinear function of ε accounting for nonlinear viscoelastic effects and given in parametric form in terms of unknown constants $d_1, d_2, \ldots$. The function f, a relaxation function of the material, will also be given in parametric form. More precisely, it will be assumed to satisfy an Nth order differential equation with constant coefficients.

Since our system will be identified by using experiments of the relaxation type, equation (1), under constant strain $\varepsilon(t) = \varepsilon_i$, $i = 1, 2, \ldots, M$, reduces to

$$\sigma_i = g(\varepsilon_i, k_1, k_2, \ldots) + h(\varepsilon_i, d_1, d_2, \ldots) \int_0^t f(\tau)\, d\tau, \tag{2}$$

a form from which we shall proceed for identification purposes. The problem is now to optimally determine the constants $k_1, k_2, \ldots, d_1, d_2, \ldots$, and the parameters that characterize the kernel $f(t)$ such as to minimize a convenient norm involving the experimental data given in relaxation form. To this end we shall divide the identification problem in two parts. First we shall determine a convenient kernel $f(t)$ using differential approximation. This is done in the next section. Then we shall proceed to optimally determine the remaining unknowns using a Gauss–Newton iterative method.

3. DIFFERENTIAL APPROXIMATION OF THE KERNEL

We shall assume that the relaxation function $f(t)$ appearing in equation (2) satisfies the differential equation of order N given by

$$a_0 f + a_1 f^{(1)} + \ldots + a_{N-1} f^{(N-1)} + f^{(N)} = 0, \tag{3}$$

where $a_0, a_1, \ldots, a_{N-1}$ are coefficients to be determined.

We additionally assume that the numerical data of relaxation furnished by the experiments has been conveniently fitted with analytical functions of the type

$$\bar{\sigma}_i = \sigma_{i0} + \sigma_{it} F_i(t, m_{i1}, m_{i2}, \ldots), \tag{4}$$

where σ_{i0}, σ_{it} and $m_{i1}, m_{i2}, \ldots$, are constants to be determined for each experiment i, by standard fitting procedures.

Comparing equations (2) and (4), we wish to optimally determine $a_0, a_1, \ldots, a_{N-1}$ in equation (3) such as $\int_0^t f(\tau)\, d\tau$ approximates functions $F_i(t, m_{i1}, m_{i2}, \ldots)$ in some sense. This is of course equivalent to make the approximation

$$f(t) \cong v_i(t), \tag{5}$$

where

$$v_i(t) = \frac{d}{dt} F_i(t, m_{i1}, m_{i2}, \ldots). \tag{6}$$

One way to treat this problem is by differential approximation [7]. Using this method we required that the functional

$$\sum_{i=1}^{M} \int_{t_1}^{t_2} [a_0 v_i + a_1 v_i^{(1)} + \ldots + a_{N-1} v_i^{(N-1)} + v_i^{(N)}]^2\, dt, \tag{7}$$

be a minimum with respect to all possible choices of the constants $a_0, a_1, \ldots, a_{N-1}$. The minimization of this quadratic functional leads to the system of N simultaneous linear equations

$$\sum_{i=1}^{M} \int_{t_1}^{t_2} [a_0 v_i + a_1 v_i^{(1)} + \ldots + a_{N-1} v_i^{(N-1)} + v_i^{(N)}] v_i^{(j)} \, dt = 0, \; j = 0, 1, \ldots, N-1, \tag{8}$$

from which the unknown coefficients $a_0, a_1, \ldots, a_{N-1}$, can be computed. Once the coefficients have been determined, we use them to solve the characteristic equation

$$a_0 + a_1 R + a_2 R^2 + \ldots + a_{N-1} R^{N-1} + R^N = 0 \tag{9}$$

and obtain for $f(t)$ the representation in terms of exponentials

$$f(t) = \sum_{j=1}^{N} C_j e^{R_j t} \tag{10}$$

Subsequently, we can express $\sigma_i(t)$ given by equation (2) in the form

$$\sigma_i = g(\varepsilon_i, k_1, k_2, \ldots) + h(\varepsilon_i, d_1, d_2, \ldots) \sum_{j=1}^{N} \frac{C_j}{R_j} (1 - e^{R_j t}) \tag{11}$$

or by defining

$$C_j^* = \frac{C_j}{R_j}, \qquad H_j(t) = 1 - e^{R_j t} \tag{12}$$

σ_i will be given by

$$\sigma_i = g(\varepsilon_i, k_1, k_2, \ldots) + h(\varepsilon_i, d_1, d_2, \ldots) \sum_{j=1}^{N} C_j^* H_j(t). \tag{13}$$

It should be noted, that the previous derivation has been done under the assumption that the roots R_j of the characteristic equation are real and distinct. Multiple and complex roots can be equally handled with obvious modifications.

4. OPTIMIZATION

Now, the identification of our system can be formulated as the following optimization problem: given M pairs of independent experimental functions $\bar{\sigma}_i(t)$ and $\bar{\varepsilon}_i$, find the constants $k_1, k_2, \ldots, d_1, d_2, \ldots, C_1^*, C_2^*, \ldots, C_N^*$, such that the functional

$$\mathscr{E}(k_1, k_2, \ldots, d_1, d_2, \ldots, C_1^*, C_2^*, \ldots) = \sum_{i=1}^{M} \mu_i \int_0^{t_i} [\sigma_i^*(t) - \bar{\sigma}_i(t)]^2 \, dt \tag{14}$$

is minimized, where μ_i are suitable weighting factors,

$$\sigma_i^*(t) = g(\bar{\varepsilon}_i, k_1, k_2, \ldots) + h(\bar{\varepsilon}_i, d_1, d_2, \ldots) \sum_{j=1}^{N} C_j^* H_j(t), \tag{15}$$

and $\bar{\sigma}_i(t)$ is given by equation (4).

The upper limit t_i of the integral in equation (14) denotes the duration of the ith experiment.

In order to simplify the notation, let us define

$$E_i(t) = \sigma_i^*(t) - \bar{\sigma}_i(t). \tag{16}$$

Then equation (14) may be written as follows

$$\mathscr{E}(k_1, k_2, \ldots, d_1, d_2, \ldots, C_1^*, C_2^*, \ldots) = \sum_{i=1}^{M} \mu_i \int_0^{t_i} E_i^2(t)\, \mathrm{d}t. \tag{17}$$

The minimization of the expression (17) leads to the system of N nonlinear simultaneous equations

$$\sum_{i=1}^{N} \mu_i \int_0^{t_i} E_i(t) \frac{\partial E_i(t)}{\partial \alpha}\, \mathrm{d}t = 0, \alpha = k_1, k_2, \ldots, d_1, d_2, \ldots, C_1^*, C_2^*, \ldots, C_N^*. \tag{18}$$

5. SOLUTION METHOD

In order to solve the nonlinear optimization problem formulated in the last section, we shall use an extension of the Gauss–Newton iterative scheme [8] as employed in [5]. To this end we expand the function $E_i(t)$ up to linear terms with respect to the unknown constants, i.e.

$$E_i(t) \cong E_i^0(t) + \sum_{m=1}^{NK} \frac{\partial g^0}{\partial k_m} \Delta k_m + h^0 \sum_{j=1}^{N} H_j^0(t) \Delta C_j^* + \sum_{j=1}^{N} C^{*0}_j H_j^0(t) \sum_{n=1}^{ND} \frac{\partial h^0}{\partial d_n} \Delta d_n \tag{19}$$

where the superscript zero indicates that the function is evaluated at the currently known values of the unknown quantities, and where

$$\begin{aligned} k_m^1 &= k_m^0 + \Delta k_m, & m &= 1, 2, \ldots, NK \\ d_n^1 &= d_n^0 + \Delta d_n, & n &= 1, 2, \ldots, ND \\ C_j^{*1} &= C_j^{*0} + \Delta C_j^*, & j &= 1, 2, \ldots, N \end{aligned} \tag{20}$$

are the improved values of the corresponding quantities.

By substitution of $E_i(t)$ given by equation (19) into equation (17), our original optimization problem reduces to the solution of a least squares problem at each iteration, i.e., to the solution of

$$\begin{aligned} &\sum_{k=1}^{M} \mu_k \sum_{j=1}^{NK} \Delta k_j \int_0^{t_k} \frac{\partial g^0}{\partial k_j} \frac{\partial g^0}{\partial k_i}\, \mathrm{d}t + \sum_{k=1}^{M} \mu_k \sum_{j=1}^{N} \Delta C_j^* \int_0^{t_k} h^0 H_j^0 \frac{\partial g^0}{\partial k_i}\, \mathrm{d}t \\ &\quad + \sum_{k=1}^{M} \mu_k \sum_{j=1}^{ND} \Delta d_j \int_0^{t_k} \left[\sum_{l=1}^{N} C_l^{*0} H_l^0 \right] \frac{\partial h^0}{\partial d_j} \frac{\partial g^0}{\partial k_i}\, \mathrm{d}t \\ &\quad = \sum_{k=1}^{M} \mu_k \int_0^{t_k} \left[\bar{\sigma}_k - g^0 - h^0 \sum_{l=1}^{N} C_l^{*0} H_l^0 \right] \frac{\partial g^0}{\partial k_i}\, \mathrm{d}t, \quad i = 1, 2, \ldots, NK, \\ &\sum_{k=1}^{M} \mu_k \sum_{j=1}^{NK} \Delta k_j \int_0^{t_k} \frac{\partial g^0}{\partial k_j} h^0 H_i^0\, \mathrm{d}t + \sum_{k=1}^{M} \mu_k \sum_{j=1}^{N} \Delta C_j^* \int_0^{t_k} h^0 H_j^0 h^0 H_i^0\, \mathrm{d}t \\ &\quad + \sum_{k=1}^{M} \mu_k \sum_{j=1}^{ND} \Delta d_j \int_0^{t_k} \left[\sum_{l=1}^{N} C_l^{*0} H_l^0 \right] \frac{\partial h^0}{\partial d_j} h^0 H_i^0\, \mathrm{d}t \\ &\quad = \sum_{k=1}^{M} \mu_k \int_0^{t_k} \left[\bar{\sigma}_k - g^0 - h^0 \sum_{l=1}^{N} C_l^{*0} H_l^0 \right] h^0 H_i^0\, \mathrm{d}t, \qquad i = 1, 2, \ldots, N, \end{aligned} \tag{21}$$

$$\sum_{k=1}^{M} \mu_k \sum_{j=1}^{NK} \Delta k_j \int_0^{t_k} \frac{\partial g^0}{\partial k_j} \left[\sum_{l=1}^{N} C_l^{*0} H_l^0 \right] \frac{\partial h^0}{\partial d_i} \, dt + \sum_{k=1}^{M} \mu_k \sum_{j=1}^{N} \Delta C_j^* \int_0^{t_k} h^0 H_j^0 \left[\sum_{l=1}^{N} C_l^{*0} H_l^0 \right] \frac{\partial h^0}{\partial d_i} \, dt$$
$$+ \sum_{k=1}^{M} \mu_k \sum_{j=1}^{ND} \Delta d_j \int_0^{t_k} \left[\sum_{l=1}^{N} C_l^{*0} H_l^0 \right]^2 \frac{\partial h^0}{\partial d_j} \frac{\partial h^0}{\partial d_i} \, dt$$
$$= \sum_{k=1}^{M} \mu_k \int_0^{t_k} \left[\bar{\sigma}_k - g^0 - h^0 \sum_{l=1}^{N} C_l^{*0} H_l^0 \right] \left[\sum_{l=1}^{N} C_l^{*0} H_l^0 \right] \frac{\partial h^0}{\partial d_i} \, dt, \qquad i = 1, 2, \ldots, ND$$

(21) *cont.*

a linear system of equations in terms of the $NK+N+ND$ quantities Δk_i, ΔC_j^*, Δd_k, $i = 1, \ldots, NK$, $j = 1, \ldots, N$ and $k = 1, \ldots, ND$. After this system has been solved, we compute the new estimates of the parameters by using equations (20) and the procedure is continued until convergence is reached. This process is quadratically convergent near the solution. Obviously, the first step of the process requires an *a priori* estimate of the unknown parameters.

6. PREDICTION: DIRECT AND INVERSE PROBLEM

In the last sections we have optimally determined a number of constants that uniquely define the model equation given by (1). We wish to show now how to solve equation (1) in the sense of

(a) given a certain strain history ε, find the associated stress history σ.

(b) given a certain stress history σ, find the strain ε.

We can see that problem (b) contains as a special case, the determination of the creep function. We shall specifically refer to that aspect of the problem in Sections 7 and 8, where numerical examples will be presented.

Problem (a) is the simplest of the two. For a given input ε, we can readily compute σ by direct substitution in the model equation (1). However, we observe that this procedure involves the computation of a convolution, a very inefficient operation insofar as time and storage is concerned. Fortunately we can bypass this difficulty by reducing the model equation (1) to a system of nonlinear ordinary differential equations subject to initial conditions. To this end, we substitute f given by equation (10) in equation (1) obtaining

$$\sigma = g(\varepsilon(t)) + \sum_{i=1}^{N} C_i Z_i(t), \tag{22}$$

where $Z_i(t)$, $i = 1, 2, \ldots, N$, are functions defined by

$$Z_i(t) = e^{R_i t} \int_0^t h(\varepsilon(\tau)) \, e^{-R_i \tau} \, d\tau. \tag{23}$$

By differentiating equation (23) with respect to t, we find that Z_i satisfies the following system of differential equations

$$\frac{dZ_i}{dt} - R_i Z_i = h(\varepsilon(\tau)), \qquad Z_i(0) = 0, \qquad i = 1, \ldots, N, \tag{24}$$

subject to initial conditions, a very convenient device insofar as economy in computer storage is concerned.

In g and h appearing in equations (22) and (24) we have omitted the explicit dependence of these functions with the material constants, in an effort to simplify the notation.

Clearly, for a given ε we can integrate equations (24) and obtain the Z_i's that substituted in equation (22) directly furnishes the required value of σ, thus disposing of problem (a).

The solution of problem (b), that along classical lines is considerably more complicated than problem (a), in the present formulation requires only very little additional effort. In fact, assuming that function $g(\varepsilon)$ appearing in equation (22) possesses an inverse i.e.,

$$g(\varepsilon) = q \leftrightarrow \varepsilon = G(q) \tag{25}$$

then, we can invert equation (22) in the sense of

$$\varepsilon = G\left(\sigma - \sum_{i=1}^{N} C_i Z_i(t)\right), \tag{26}$$

a function that substituted in equation (24) yields

$$\frac{\mathrm{d}Z_i}{\mathrm{d}t} = R_i Z_i + h\left[G\left(\sigma(t) - \sum_{i=1}^{N} C_i Z_i(t)\right)\right], \qquad Z_i(0) = 0, \qquad i = 1, \ldots, N, \tag{27}$$

a system of nonlinear differential equations of the first order in Z_i, subject to initial conditions. Note that ε follows from equation (26).

The success of this method, i.e., the inversion of nonlinear integral equations of the Volterra type by reduction to differential systems is mainly due to the expansion of the kernel in exponential functions and in the possibility of inverting the nonlinear function $g(\varepsilon)$. If this inversion cannot be analytically performed, i.e. if G in equation (25) does not afford an analytical expression, we can always perform the inversion in a numerical fashion.

We observe that the solution of problems (a) and (b) require only to integrate systems of first order, ordinary differential equations subject to initial conditions, a task for which many standard algorithms and computer routines are available. But what it is important to remark is that this process requires only the storage of the computer program and the quantities currently being computed, an insignificant number in comparison with the storage of the whole functions if a quadrature approach to the convolutions is used.

7. RELAXATION OF SOLID POLYURETHANE

In order to make a specific application of the present method of identification and prediction, we present a numerical example utilizing experimental data on solid polyurethane furnished by W. N. Findley and J. S. Lai in [9] and [10]. To this end we consider that functions g and h appearing in the model equation (1) are given by

$$g(\varepsilon, k_1, k_2) = k_1\varepsilon + k_2\varepsilon^2, \tag{28}$$

$$h(\varepsilon, d) = \varepsilon\, \mathrm{e}^{d\varepsilon}. \tag{29}$$

The choice of functions g and h given by equations (28) and (29) is guided by a qualitative knowledge on the instantaneous and time-dependent behavior of the material. Although there is no fixed rule to decide among a large number of functions that might satisfy the required qualitative conditions, we are generally inclined to pick functions exhibiting a simple structure and a relatively low number of constants to determine.

Under these assumptions equation (2) reduces to

$$\sigma_i(t) = k_1\varepsilon_i + k_2\varepsilon_i^2 + \varepsilon_i\, e^{d\varepsilon_i}\int_0^t f(\tau)\,d\tau. \tag{30}$$

The problem consists now in the determination of k_1, k_2, d and function f using the procedure described in Sections 3, 4 and 5, such as to satisfy the least squares criterion given by equation (14), for appropriate given data.

In reference [10] it has been shown that the relaxation stress, i.e., the stress obtained under constant strain, for solid polyurethane in tension, can be given as functions of time t of the following form

$$\bar{\sigma}_i = \sigma_{i0} - \sigma_{it}t^m, \qquad i = 1, 2, 3, 4, \tag{31}$$

where i denotes different tests at different constant strains ε_i, $i = 1, 2, 3, 4$. The values of the constants appearing in equation (31), as reported in reference [10], are given below in Table 1.

TABLE 1

i	ε_i [10^{-3} in/in]	σ_{io} [ksi]	σ_{it} [ksi/hr^m]	m
1	4·333	2·28	0·206	0·125
2	6·500	3·38	0·340	0·125
3	8·666	4·45	0·518	0·125
4	10·833	5·50	0·701	0·125

We observe that the exponent m in equation (31) is the same for each experiment. This is not a requirement of our method since all we need is an analytical expression for $\bar{\sigma}_i$ in terms of t, regardless the nature of this function. As a matter of fact, better agreement between experiments and functions (31) could be obtained by making m to be i-dependent. We have used equations (31) however in an attempt to make comparisons with the results reported in Ref. [10].

With this experimental information we proceed first to determine the coefficients a_i, $i = 0, 1, \ldots, N-1$ of the differential equation

$$a_0 f + a_1 f^{(1)} + \ldots + a_{N-1} f^{(N-1)} + f^{(N)} = 0, \tag{32}$$

using the method outlined in Section 3.

By comparing equation (31) with (4) we conclude that

$$F_i(t, m_{i1}, m_{i2}, \ldots) = -t^m. \tag{33}$$

We see that in fact F_i is independent of i, as previously noted. The values of $v_i^{(j)}$ needed in equation (8) have been computed using equations (33) and (6), i.e.,

$$v_i^{(j)} = \frac{d^{j+1}F_i}{dt^{j+1}} = -m(m-1)\ldots(m-j)t^{m-1-j}, \qquad j = 0, 1, \ldots, N-1. \tag{34}$$

The integration of equation (8) has been performed using $t_1 = 0{\cdot}05$ and $t_2 = 2{\cdot}0$, the duration of the experiments. The singularity of t^α for $\alpha < 0$, prevents the use of $t_1 = 0$.

Once the coefficients a_i, $i = 0, 1, \ldots, N-1$, have been computed, we proceed to find the N roots of the characteristic equation

$$a_0 + a_1 R + \ldots + a_{N-1} R^{N-1} + R^N = 0. \tag{35}$$

The results of these computations are presented in Table 2 for $N = 2, 3, \ldots, 9$.

Using the roots of the characteristic equation (35) we can write $f(t)$ in the form

$$f(t) = \sum_{j=1}^{N} C_j e^{R_j t}, \tag{36}$$

that substituted in equation (30) yields

$$\sigma_i(t) = k_1 \varepsilon_i + k_2 \varepsilon_i^2 - \varepsilon_i e^{d\varepsilon_i} \sum_{j=1}^{N} C_j^* (1 - e^{R_j t}), \tag{37}$$

where

$$C_j^* = \frac{C_j}{R_j}.$$

The determination of the additional unknown constants of the model can be now performed using the method outlined in Section 5.

We have used simultaneously the four experiments σ_i, $i = 1, 2, 3, 4$, given in Table 1, to optimally determine the unknown constants k_1, k_2, d, C_1^*, $C_2^*, \ldots, C_N^*$. In Table 3 we present the results of this optimization for $N = 2, 3, \ldots, 9$. In all our numerical work we have assumed $\mu_k = 1$, $k = 1$ to 5.

Additional numerical experimentation has been made using the model

$$g(\varepsilon, k_1, k_2, k_3) = k_1 \varepsilon + k_2 \varepsilon^2 + k_3 \varepsilon^3, \tag{38}$$

for the instantaneous response. Since the consequences of this change are reflected at the fourth digit of the predicted stress, we have retained only the model with k_1 and k_2 for our subsequent purposes.

Now, we are in a position to test the predictive ability of our model. To this end we have plotted σ_i given by equation (37) using the constants previously determined, for various values of N. These results may be compared with the experimental ones and with those given in Ref. [10], in Fig. 1. We observe that for $N = 4$ the model predicts very accurately the experimental values. In fact, the difference between experimental and predicted values is hardly noticeable in the figure. The predicted values for a lower order model ($N = 2$) are also shown in the same figure. Although they do not match exactly the experimental curves, are accurate enough for almost any practical purpose.

The mean square errors, calculated with $\sum_{k=1}^{R} (\sigma_{ik}^{\mathrm{PRED}} - \sigma_{ik}^{\mathrm{EXP}})^2 / R$ are given in Table 4 for each experiment and different values of N. It can be seen that for $N = 4$ or 5, the mean square error reaches a minimum value.

8. CREEP OF SOLID POLYURETHANE

Using the model constants optimally determined in the previous section, we can now proceed to use the model for prediction purposes in a number of ways. In this section we show the results of inverting the integral model in order to determine the creep function of the material.

TABLE 2

		N							
j	R_j [1/hr]	2	3	4	5	6	7	8	9
1	R_1	−3·0219	−1·6834	−1·1298	−0·84360	−0·67342	−0·56196	−0·48353	−0·42422
2	R_2	−33·576	−16·996	−10·539	−7·3258	−5·4888	−4·3361	−3·5618	−3·0051
3	R_3		−67·034	−39·261	−26·378	−19·186	−14·736	−11·780	−9·6863
4	R_4			−104·40	−66·735	−47·419	−35·806	−28·200	−22·868
5	R_5				−144·15	−97·728	−72·308	−56·220	−45·152
6	R_6					−185·53	−131·26	−100·12	−79·544
7	R_7						−228·12	−166·70	−130·07
8	R_8							−271·62	−203·45
9	R_9								−315·63

TABLE 3

		N							
j	C_j^* [10^2 ksi]	2	3	4	5	6	7	8	9
1	C_1^*	0·18412	0·16456	0·15536	0·15525	0·15639	0·15912	0·16113	0·16390
2	C_2^*	0·034110	0·031970	0·061935	0·063486	0·067844	0·064797	0·064472	0·061579
3	C_3^*		0·082712	0·018956	0·042433	0·019912	0·035497	0·030412	0·040186
4	C_4^*			0·066413	−0·012445	0·072170	0·010864	0·050446	0·010911
5	C_5^*				0·079161	−0·074978	0·083400	−0·063083	0·078251
6	C_6^*					0·10757	−0·11223	0·19977	−0·15486
7	C_7^*						0·12388	−0·23245	0·33662
8	C_8^*							0·16901	−0·35337
9	C_9^*								0·20868
k_1[10^2 ksi]		5·2165	5·2583	5·2652	5·2750	5·2829	5·2877	5·2922	5·2952
k_2[10^4 ksi]		−0·45229	−0·43590	−0·43336	−0·43009	−0·42753	−0·42599	−0·42455	−0·42361
d[10^2]		0·37143	0·35819	0·35608	0·35260	0·34982	0·34814	0·34658	0·34555

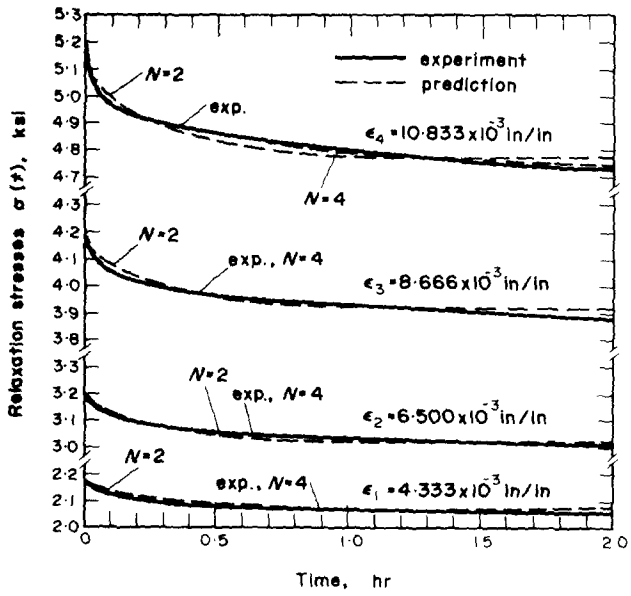

FIG. 1.

Since in the present case function g appearing in equation (25) is given by

$$g(\varepsilon) = k_1\varepsilon + k_2\varepsilon^2 = q, \tag{39}$$

we can readily invert obtaining

$$\varepsilon = G(q) = [-k_1 \pm \sqrt{(k_1^2 + 4qk_2)}]/2k_2 \tag{40}$$

where only the plus sign will be retained. Therefore, recalling equation (29), equation (27) reduces to

$$\frac{dZ_i}{dt} = R_iZ_i + G\left(\sigma - \sum_{i=1}^{N} C_iZ_i\right) \exp\left[dG\left(\sigma - \sum_{i=1}^{N} C_iZ_i\right)\right], \qquad Z_i(0) = 0, \qquad i = 1, \ldots, N \tag{41}$$

where G appearing in equation (41) is given by equation (40). Clearly ε is determined by means of

$$\varepsilon = G\left(\sigma - \sum_{i=1}^{N} C_iZ_i\right). \tag{42}$$

TABLE 4

i	ε_i [10^{-3} in/in]	Mean square error [10^{-5}]							
		$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$	$N=9$
1	4·333	3·02	0·72	0·47	0·55	0·66	0·74	0·82	0·87
2	6·500	11·39	4·67	3·91	4·00	4·18	4·32	4·46	4·56
3	8·666	26·72	10·99	9·14	9·26	9·62	9·89	10·18	10·38
4	10·833	38·59	6·20	2·24	1·60	1·52	1·55	1·62	1·70
Total mean square error		79·72	22·58	15·76	15·41	15·98	16·50	17·08	17·51

Using

$$\sigma_1 = 2{\cdot}0 \text{ ksi}$$

$$\sigma_2 = 3{\cdot}0 \text{ ksi}$$

$$\sigma_3 = 4{\cdot}0 \text{ ksi}$$

$$\sigma_4 = 5{\cdot}0 \text{ ksi}$$

we have integrated the system of equations (41) and obtained the corresponding creep strains $\varepsilon_i(t)$, $i = 1, 2, 3, 4$, by means of equation (42). These results have been plotted in Fig. 2 and compared with those experimental values for the creep strain furnished in Ref. [9]. We must note that the agreement of the theoretical and experimental values of the creep strains obtained using the model equation (1), are of the same order of magnitude that the agreement between theoretical and experimental values of the relaxation stresses reported by J. S. Lai and W. H. Findley in Ref. [9] using their model with multiple integrals.

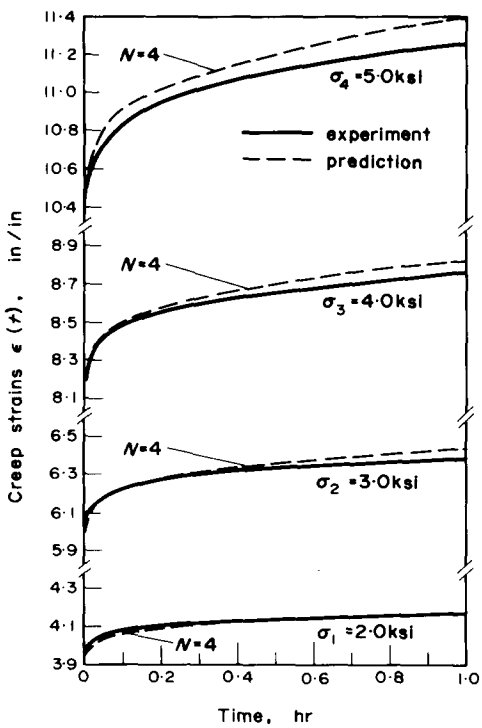

FIG. 2.

9. PREDICTION FOR A GIVEN STRAIN

We turn now to a more general prediction problem and that is: we wish to predict σ for any arbitrary given strain history $\varepsilon(t)$. To this end we integrate the system of equations (22)–(24) and determine σ using equation (22). In order to check the accuracy of the model

we consider the following strain inputs

$$\varepsilon_1 = [4{\cdot}333H(t)+4{\cdot}333H(t-2)-4{\cdot}333H(t-3)]\times 10^{-3}\ \text{in/in}$$

$$\varepsilon_2 = [10{\cdot}833H(t)-4{\cdot}333H(t-2)]\times 10^{-3}\ \text{in/in}$$

where $H(t-t_i)$ is the Heaviside unit function, for which we possess the experimental stresses given in Ref. [10]. The predicted and experimental stresses are shown in Figs. 3 and 4. In the same figures we report the results obtained with the model of multiple integrals given in [10].

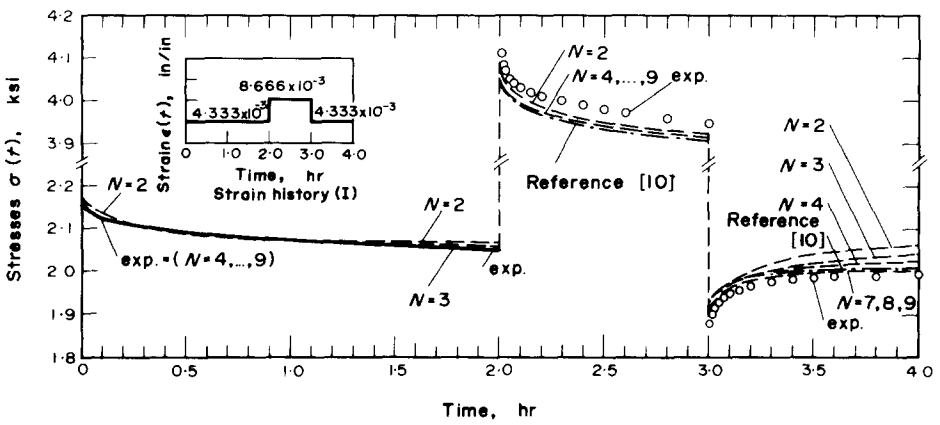

FIG. 3.

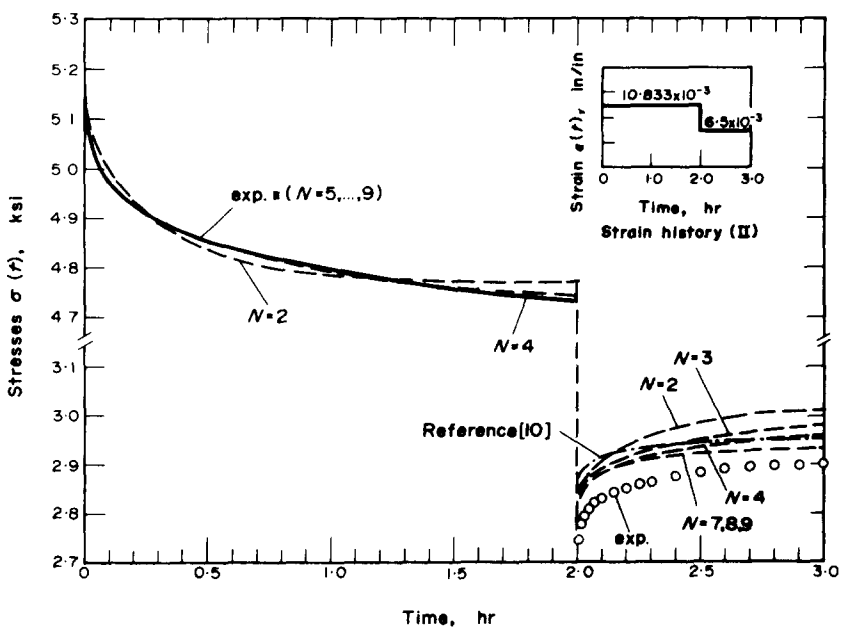

FIG. 4.

The experimental values for the first two hours were plotted using the formula given in [10], while those values for $t > 2$ h have been directly measured and plotted from the figure appearing in the same reference.

The model for $N = 7$ reaches the best agreement with the experimental values. For $N > 7$, the improvement is not appreciable. In fact, the plotted values for $N = 8$ and $N = 9$ are indistinguishable from those corresponding to $N = 7$, in Figs. 3 and 4.

10. PREDICTION FOR A GIVEN STRESS

If we wish to predict ε given σ, we can proceed as indicated in Section 6. In fact, for a given σ we integrate the initial value problem given by equations (26)–(27) and determine ε by means of equation (26). Unfortunately we have no experimental data to check the model under these conditions. In order to show the application of the present procedure we consider

$$\varepsilon(t) = t \tag{43}$$

in equation (1) and obtain

$$\sigma(t) = k_1 t + k_2 t^2 + e^{dt} \sum_{j=1}^{N} \frac{C_j}{(d-R_j)^2}[(d-R_j)t + e^{(R_j-d)t} - 1]. \tag{44}$$

Now, using σ given by equation (44) we integrated equations (26) and (27) and obtained values of ε given by equation (26) that coincided with $\varepsilon = t$ given by equation (43) in more than eight digits, establishing the high accuracy and stability of the method.

CONCLUDING REMARKS

A method to optimally determine the material constants of nonlinear viscoelastic materials characterized by equations of the type (1), using exclusively relaxation tests at various strain levels, has been presented. Extensive numerical experimentation has shown the accuracy and stability of the method. The question of the predictive ability of the model under variable stresses or strains has been examined using the experimental data given in [9], [10]. The results using the simple model (1) are very satisfactory and never inferior to those obtained by Findley and Lai using a multiple integral representation. It is expected that consideration of more general strain histories in the identification procedure, instead of the rather limited relaxation tests used here, will appreciably improve the overall predictive ability of model (1). Results of this type will be presented in a forthcoming publication.

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Абстракт—В работе используется нелинейное интегральное уравнение Вольтерры для изображения класса нелинейных, вязкоупругих материалов, подверженных действию одноосных эффектов. Определяется, оптимально, некоторое число неизвестных параметров, которые появляются в математической структуре модели, в целью сведения к минимуму функционала найменьших квадратов, из всех экспериментальных данных, приобрегая, что они получены опытами реляксации, выполненными для разных уровней деформации. Проверяется предсказанная способность модели, пользуясь экспериментальными данными, представленными в (10). В заключении, сравниваются результаты с такими же, предложенными Финдлеем и Ляи, которые пользовались многократной интегральной моделью (9), (10) и с результатами из экспериментов.